Computational Methods in Finite Geometry

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Topic # 4

Cubic Surfaces

Cubic Surfaces with 27 Lines

A cubic surface in $\mathrm{PG}(, q)$ is defined by a homogeneous cubic polynomial in 4 variables.

Cayley (1849): A smooth cubic surface has 27 lines.

We are interested in classifying smooth cubic surfaces over small finite fields.

We begin with some background material about cubic surfaces in general.

Cubic Surfaces with 27 Lines

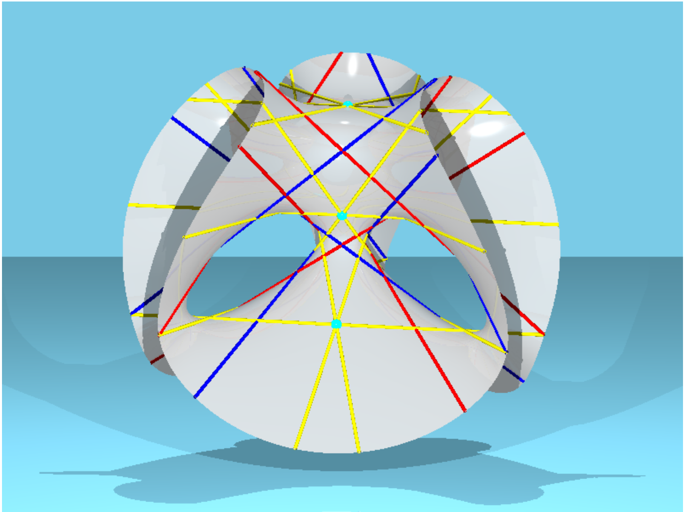
Let us investigate the geometry of cubic surfaces with 27 lines.

Example:
The Clebsch surface.
The picture shows the affine part of the surface with equation

$$\begin{aligned}
&-3+9(x+y+z)+3(x^2+y^2+z^2)\\
&-42(xy+xz+yz)\\
&-9(x^3+y^3+z^3)\\
&+21(x^2y+x^2z+xy^2+xz^2+y^2z+yz^2)\\
&-6xyz=0.
\end{aligned}$$

The equation was chosen so that all 27 lines are real.

Cubic Surfaces with 27 Lines



The picture is based on earlier work by Alan Esculier.

Animation!

Tritangent Planes

Intersect the surface with a plane.

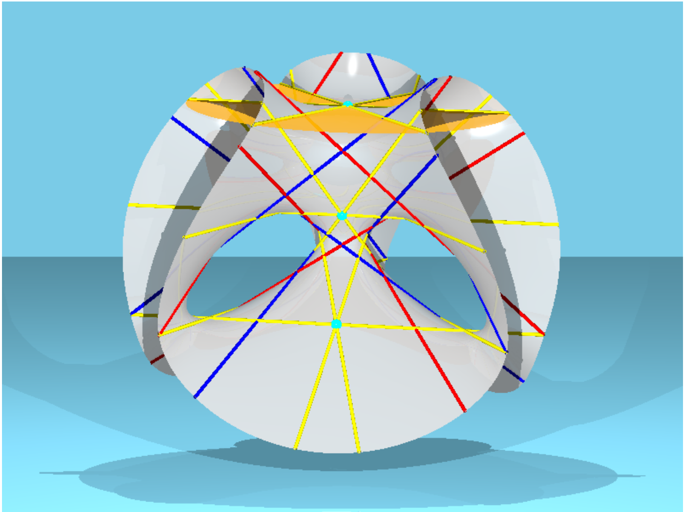
The cubic surface induces a cubic curve on that plane.

That curve may degenerate into three lines.

Such a plane is called a tritangent plane.

Here is an example:

Tritangent Planes



Animation!

Why 27 lines?

Proof (following Salmon)
Let $\{f = 0\}$ be the equation of the surface. Assume there is at least one line.

Let $\mathrm{PG}(3, q)$ be defined on the four-dimensional vector space $\mathbb{F}^4$ with basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$, and let $[x_1 : x_2 : x_3 : x_4]$ be homogeneous coordinates for $\mathrm{PG}(3, q)$.

Since $\mathrm{PGL}(4, \mathbb{F})$ is transitive on lines of $\mathrm{PG}(3, \mathbb{F})$, we may assume that this line is $L = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \{x_3 = x_4 = 0\}$.

The equation $f = 0$ can then be written as $x_3 U + x_4 V = 0$ with U and V quadratic in the x_i.

We find all tritangent planes through L since only those planes give us lines incident with the given line:

Why 27 lines?

Any plane through L can be obtained by assuming that

$$x_3 = \mu x_4$$

for some scalar μ.

Substituting this into the equation, and dividing by x_4 yields a form which is quadratic in x_1, x_2, and x_4.

The discriminant of this quadratic form is a polynomial in μ of degree 5.

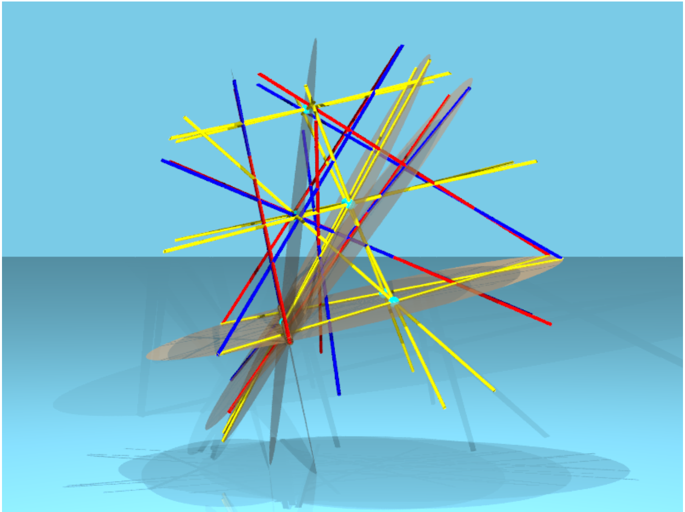
Why 27 lines?

Hence there are 5 values of μ for which a hyperplane through L is tritangent.

Now, do some counting:

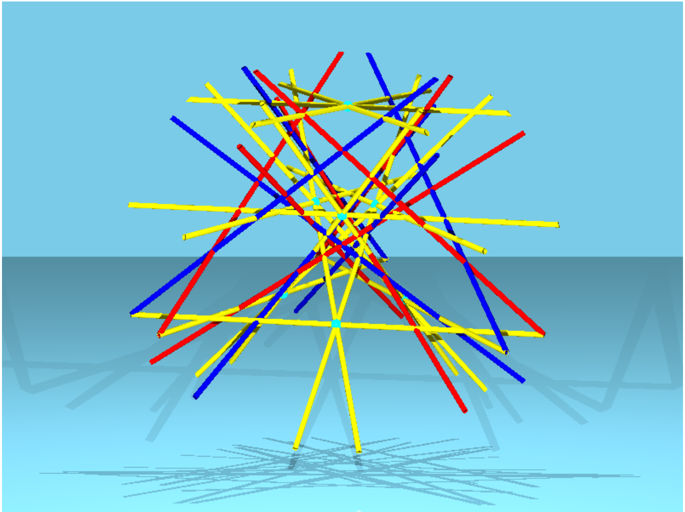
$$3 + (5 - 1) \cdot 3 \times 2 = 27.$$

Five tritangent planes through a line



Animation!

The 27 Lines



The 27 lines (surface removed) Animation!

A Double Six

Following Schläfli, we distinguish 6 lines (red), 6 firther lines (blue) and 15 lines (yellow).

The 6 red lines $a_1, a_2, \ldots, a_6$ and the 6 blue lines $b_1, b_2, \ldots, b_6$ form a double six.

It is customary to indicate a double six in the following notation:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$$

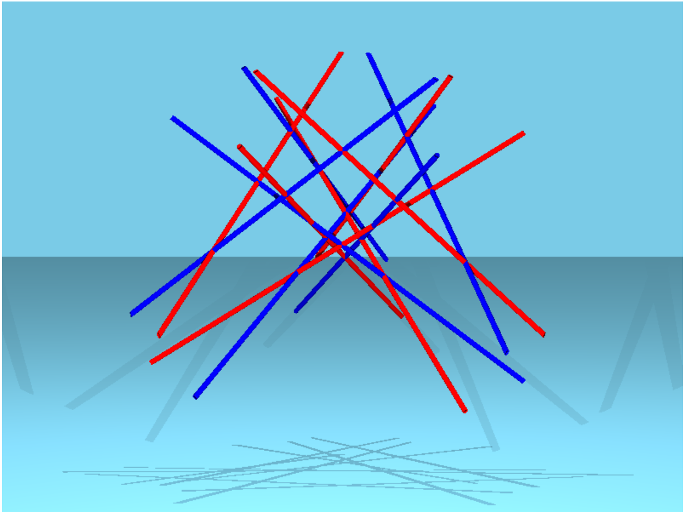
A Double Six

Each line in the array intersects the lines not in the same row or column.

So, for instance, b_6 intersects exactly a_1, a_2, a_3, a_4, a_5.

There is an extra condition on the 12 lines that we will explain below.

A Double Six



A double six Animation!

The c_{ij} lines

Fifteen further lines are defined using the formula

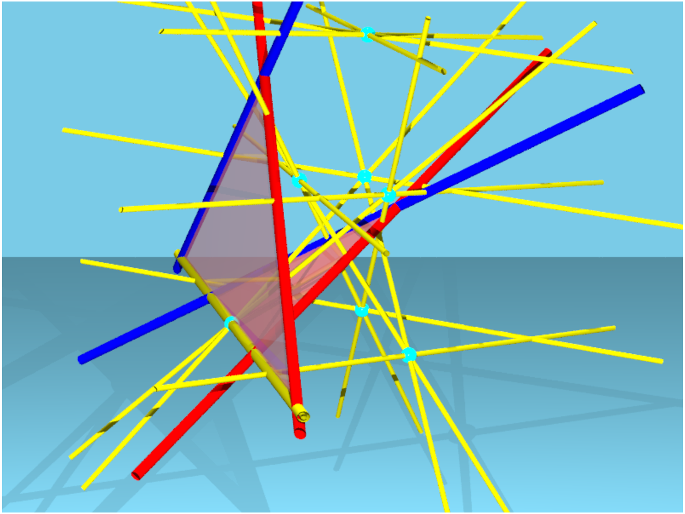
$$c_{ij} = a_i b_j \cap a_j b_i.$$

Here, $a_i b_j$ is the tritangent plane spanned by a_i and b_j.

Likewise, $a_j b_i$ is the tritangent plane spanned by a_j and b_j.

These fifteen lines c_{ij} are drawn in yellow.

The c_{ij} lines



The line $c_{14} = a_1 b_4 \cap a_4 b_1$ Animation!

The extra condition

There is an extra condition on the 12 lines of a double six:

For each j, each set of 4 of the $\{a_1, \ldots, a_6\} \setminus \{a_j\}$ has a unique, distinct, second common transversal distinct from b_j.

Notation: If the 4 lines are $\{a_1, \ldots, a_6\} \setminus \{a_j, a_i\}$, the second transversal is called b_i.

The extra condition

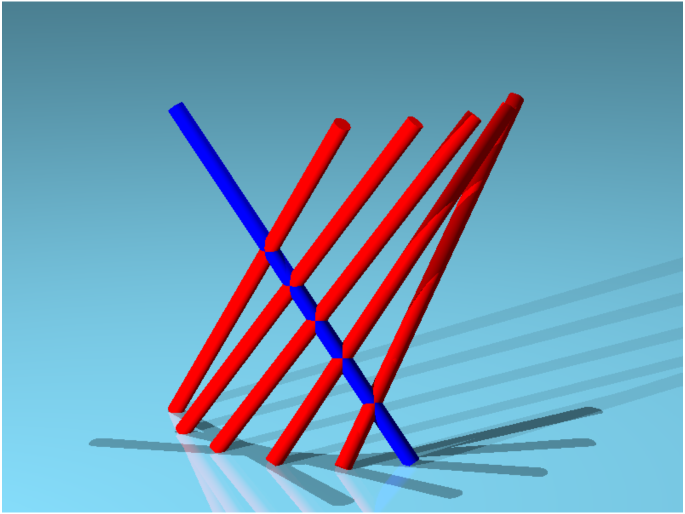
If we don’t impose the extra condition, things can go wrong:

We may have 5 lines of a regulus.

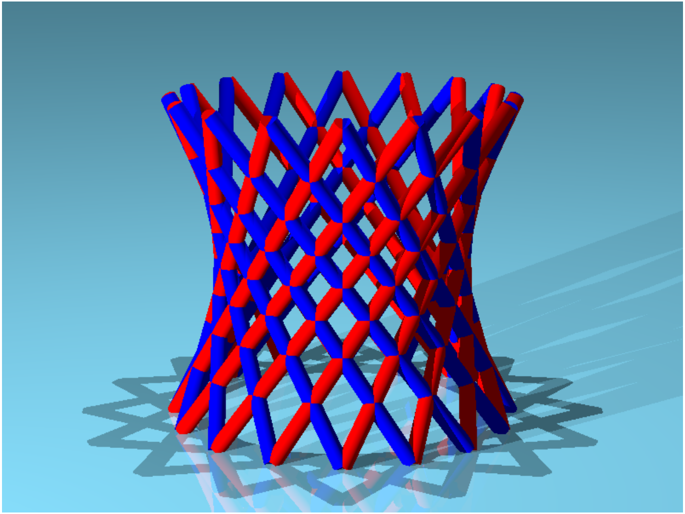
Then any 4 determine a large number of common transversals.

We don’t want this.

Violating the extra condition



Violating the extra condition



Schläfli's Theorem

Schläfli 1858

Given five skew lines a_1, a_2, a_3, a_4, a_5 with a single transversal b_6 such that each set of four a_i omitting a_j ($j = 1, \ldots, 5$) has a unique further transversal b_j, then the five lines b_1, b_2, b_3, b_4, b_5 also have a transversal a_6.

Schläfli's Theorem

Observe:
The number of monomials of degree 3 in 4 variables is 20:

$$\begin{array}{rr}
X^3 & XZ^2 \\
Y^3 & YZ^2 \\
Z^3 & Z^2W \\
W^3 & XW^2 \\
X^2Y & YW^2 \\
X^2Z & ZW^2 \\
X^2W & XYZ \\
XY^2 & XYW \\
Y^2Z & XZW \\
Y^2W & YZW
\end{array}$$

So, cubic surfaces live in $\mathrm{PG}(19, q)$.

Forcing a line to lie on the surface implies 4 conditions (if a cubic is forced to be zero on 4 distinct points of a line, it vanishes on the line).

Schläfli's Theorem

A Schläfli double six determines a unique cubic surface with 27 lines.

Proof:
The five lines $a_1, \ldots, a_5$, already impose 19 conditions:

The transversal b_6 gives 4 conditions.

Each a_i intersects b_6 and hence gives at most 3 more conditions.

The generality condition makes sure that these are independent, so

$$4 + 5 \cdot 3 = 19.$$

Classification

The following algorithm produces a classification of cubic surfaces with 27 lines.

It also produces the associated automorphism groups.

We apply the LEMMA.

We will do two attempts.

Here is attempt 1:

Classification – Attempt 1

Let $G = \mathrm{P\Gamma L}(4, q)$.

Let $\mathcal{A}$ be the set of double sixes in $\mathrm{PG}(3, q)$ (the substructures)

Let $\mathcal{B}$ be the set of cubic surfaces in $\mathrm{PG}(3, q)$

Let $\mathfrak{R}$ be the inclusion relation.

Classification – Attempt 1

It is known that every cubic surface with 27 lines has exactly 36 double sixes of lines on it.

So, we have the following algorithm:

Classify the double sixes in $\mathrm{PG}(3, q)$: $D_1, \ldots, D_m$ (the a_i in the LEMMA).

Compute the associated cubic surfaces: Let S_i be the surface determined by D_i.

Initialize the stabilizer of S_i with the stabilizer of D_i.

Let $\mathfrak{T}$ be the set $\{(D_1, S_1), \ldots, (D_m, S_m)\}$.

Classification – Attempt 1

While $\mathfrak{T} \neq \emptyset$

Pick the lexicographically least element in $\mathfrak{T}$, say (D_i, S_i).

Remove (D_i, S_i) from $\mathfrak{T}$.

Compute the 36 double sixes $E_1, \ldots, E_{36}$ associated with S_i.

Classification – Attempt 1

For each $j = 1, \ldots, 36$, perform constructive recognition to determine the index $a \leq m$ and a group element g such that

$$E_j^g = D_a.$$

If $a \neq i$, and if $(D_a, S_a) \in \mathfrak{T}$, remove (D_a, S_a) from the list $\mathfrak{T}$.

Otherwise, store g as a new generator for the stabilizer of S_i.

Classification – Attempt 2

In Attempt 2, we replace the substructure:

Let $\mathcal{A}$ be the set of five sufficiently general lines in $\mathrm{PG}(3, q)$ with a common tranversal (the substructures)

Let $\mathcal{B}$ be the set of cubic surfaces in $\mathrm{PG}(3, q)$

Let $\mathfrak{R}$ be the inclusion relation.

Classification – Attempt 2

The only major change is that each surface has $432 = 36 \cdot 12$ elements of $\mathcal{A}$ on it.

This is because each double 6 gives rise to 12 configurations of 5 lines with a common transversal.

The common transversal can be picked in 12 ways from the 12 lines of a double six.

If the common transversal is known, the 5 lines must be the lines not in the same row or column in the double six.

Classification – Attempt 2

It remains to classify five lines in $\mathrm{PG}(3, q)$ with a common transversal.

This can be done on the Klein quadric.

Suppose we call the five lines $a_1, \ldots, a_5$ and we let b_6 be the common transversal.

Let κ be the Klein correspondence

$$\text{lines in } \mathrm{PG}(3, q) \leftrightarrow \text{points of } Q(5, q)$$

Classification – Attempt 2

Recall that two lines a, b in $\mathrm{PG}(3, q)$ intersect if and only if $\kappa(a)$ and $\kappa(b)$ are collinear in a line of the quadric.

Since we want lines which are pairwise disjoint, we need them to form a partial ovoid, i.e. a coclique in the collinearity graph of the quadric.

The $\kappa(a_i)$ for $i = 1. \ldots, 5$ form a partial ovoid in $\kappa(b_6)^\perp$.

Classification – Attempt 2

Let G be the subgroup of the stabilizer of the Klein quadric corresponding to $\mathrm{P\Gamma L}(4, q)$.

G is transitive on points, so we can choose $\kappa(b_6)$ to be any point P_0, say.

The partial ovoids of size 5 in the tangent cone of P_0 can then be classified under the group

$$\mathrm{Stab}_G(P_0).$$

Classification – Attempt 2

Theorem (B., unpublished)

The cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$ are classified for $q \leq 97$.

q	#
4	1
7	1
8	1
9	2
11	2
13	4
16	5
17	7

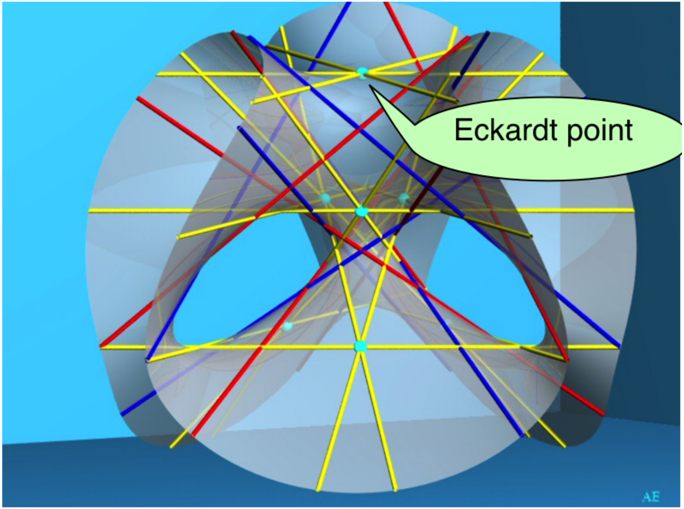
q	#
19	10
23	16
25	18
27	11
29	34
31	43
32	11
37	77

q	#
41	107
43	126
47	169
49	121
53	258
59	376
61	427
64	101

q	#
67	595
71	731
73	813
79	1081
81	331
83	1292
89	1673
97	2304

Eckardt Points

An Eckardt Point is a point where three lines of the surface intersect:



Eckardt Points

Let $\#E$ be the number of Eckardt points on a given surface.

It is known that

$$0 \leq \#E \leq 45.$$

Eckardt Points

Here is the classification of cubic surfaces with 27 lines in $\mathrm{PG}(3, q)$ by Eckardt points:

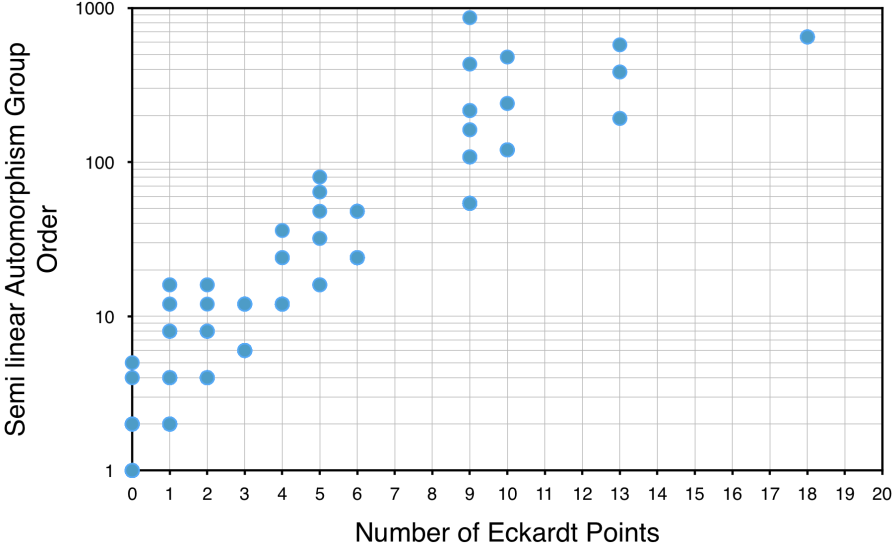
q	0	1	2	3	4	5	6	9	10	13	18	45
4	0	0	0	0	0	0	0	0	0	0	0	**1**
7	0	0	0	0	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	0	0	1	0	0
9	0	0	0	0	0	0	0	1	1	0	0	0
11	0	0	0	0	0	0	1	0	1	0	0	0
13	0	0	0	0	**1**	0	**1**	**1**	0	0	**1**	0
16	0	0	0	1	0	1	0	1	0	1	0	**1**
17	0	**1**	0	1	**2**	0	**3**	0	0	0	0	0
19	0	0	2	2	1	0	2	1	1	0	1	0
23	0	2	2	4	3	0	5	0	0	0	0	0
25	0	4	3	3	2	0	3	2	0	0	1	0

q	0	1	2	3	4	5	6	9	10	13	18	45
27	0	2	2	2	2	0	2	1	0	0	0	0
29	1	6	7	11	3	0	5	0	1	0	0	0
31	1	10	9	11	3	0	5	2	1	0	1	0
32	1	3	0	4	0	2	0	0	0	1	0	0
37	4	25	14	18	5	0	7	3	0	0	1	0
41	9	37	19	28	5	0	8	0	1	0	0	0
43	11	48	21	27	6	0	9	3	0	0	1	0
47	20	67	26	38	7	0	11	0	0	0	0	0
49	16	46	19	25	4	0	6	3	1	0	1	0
53	40	110	36	52	8	0	12	0	0	0	0	0
59	72	166	48	68	8	0	13	0	1	0	0	0

q	0	1	2	3	4	5	6	9	10	13	18	45
61	85	193	53	69	8	0	12	5	1	0	1	0
64	20	51	0	17	0	7	0	2	0	3	0	**1**
67	139	275	65	85	10	0	15	5	0	0	1	0
71	189	335	75	105	10	0	16	0	1	0	0	0
73	216	378	80	105	11	0	16	6	0	0	1	0
79	321	500	97	127	11	0	17	6	1	0	1	0
81	100	149	30	38	4	0	6	3	1	0	0	0
83	411	592	107	149	13	0	20	0	0	0	0	0
89	577	759	127	176	13	0	20	0	1	0	0	0
97	868	1033	154	203	15	0	22	8	0	0	1	0

Results:

The distribution of automorphism groups of these surfaces with respect to the number of Eckardt points is:



Semi linear Automorphism Group Order
1000
100
10
1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
Number of Eckardt Points

A Family of Cubic Surfaces

Next, we will describe a family of cubic surfaces with $\#E = 6$ invariant under Sym_4

A Family of Cubic Surfaces

Consider the 4×4 matrices

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S_3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Let s_i be the projective transformation induced by S_i.

A Family of Cubic Surfaces

Lemma

The subgroup A of $\mathrm{PGL}(4, q)$ generated by s_1, s_2, s_3 is isomorphic to Sym_4 if q is odd and isomorphic to Sym_3 if q is even.

Proof:

Verify the relations of Sym_4:

$$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = 1.$$

Note:

For q even, $S_3 = S_1$ and hence $A \simeq \mathrm{Sym}_3$.

A Family of Cubic Surfaces

Let q be odd.

For $a \notin \{0, \pm 1\}$, $a^2 \neq \pm 1$, and $b \neq 0$, consider the line

$$l_{a,b} = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{bmatrix}.$$

Let $\mathcal{O}_{a,b}$ be the orbit under A of the line $l_{a,b}$.

A Family of Cubic Surfaces

Lemma

$\mathcal{O}_{a,b}$ is a double six.

Let

$$\mathcal{S}_{a,b}$$

be the surface defined by $\mathcal{O}_{a,b}$.

A Family of Cubic Surfaces

The parameter b is not that interesting:

The normalizer of A modulo A acts regularly on the set

$$\{\mathcal{S}_{a,b} \mid b \neq 0\}$$

Info: The normalizer is generated by

$$n_\alpha = \mathrm{diag}(1, 1, 1, \alpha)$$

where α is a primitive element of $\mathbb{F}_q$.

Hence we may restrict ourselves to the surfaces

$$\mathcal{S}_a := \mathcal{S}_{a,1}\,.$$

A Family of Cubic Surfaces

Lemma

The equation of $\mathcal{S}_{a,b}$ is

$$W^3 - b^2(X^2 + Y^2 + Z^2)W + \frac{b^3}{a}(a^2 + 1)XYZ = 0.$$

Lemma

The surface $\mathcal{S}_{a,b}$ has at least 6 Eckardt points.
If $\sqrt{5} \in \mathbb{F}_q$ and $a = -2 \pm \sqrt{5}$ and $b \neq 0$, then $\mathcal{S}_{a,b}$ has at least 10 Eckardt points.
If $a = \pm\sqrt{-3} \in \mathbb{F}_q$ or $a = \pm\sqrt{-\frac{1}{3}} \in \mathbb{F}_q$ and $b \neq 0$, then $\mathcal{S}_{a,b}$ has at least 18 Eckardt points.

A Family of Cubic Surfaces

The three lines c_{12}, c_{34} and c_{56} form a triangle in the plane $W = 0$, given by the lines

$$c_{12} : \; Z = W = 0, \;\; c_{34} : \; Y = W = 0, \;\; c_{56} : \; X = W = 0.$$

Thus, $W = 0$ is a tritangent plane.

The group A acts transitively on these three lines and fixes the tritangent plane.

The double six $\mathcal{O}_{a,b}$ and the remaining 12 lines form two further orbits under A of size 12 each.

A Family of Cubic Surfaces

The tritangent plane $W = 0$ contains six Eckardt points, two on each side of the triangle:

$$\begin{aligned} E_1 &:= c_{12} \cap c_{35} \cap c_{46} = P(1, 1, 0, 0), \\ E_2 &:= c_{12} \cap c_{34} \cap c_{36} = P(1, -1, 0, 0), \\ E_3 &:= c_{34} \cap c_{16} \cap c_{25} = P(1, 0, 1, 0), \\ E_4 &:= c_{34} \cap c_{15} \cap c_{26} = P(1, 0, -1, 0), \\ E_5 &:= c_{56} \cap c_{14} \cap c_{23} = P(0, 1, 1, 0), \\ E_6 &:= c_{56} \cap c_{13} \cap c_{24} = P(0, 1, -1, 0). \end{aligned}$$

The group A acts transitively on these six Eckardt points.